

Perturbation Method in Musielak-Orlicz Sequence Spaces

Georgiev Pando^{1*†}, Zhelinski Vasil^{2†} and Zlatanov Boyan^{3†}

¹Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 Acad. Georgi Bonchev Str., Sofia, 1113, Bulgaria.

²Faculty of Mathematics and Informatics, Sofia University St. Kliment Ohridski, 5 James Bourchier Blvd., Sofia, 1504, Bulgaria.

³Faculty of Mathematics and Informatics, University of Plovdiv Paisii Hilendarski, 24 Tsar Assen Str., Plovdiv, 4000, Bulgaria.

*Corresponding author(s). E-mail(s): pandogeorgiev2020@gmail.com;
Contributing authors: vasil_zhelinski@uni-plovdiv.bg;
bobbyz@uni-plovdiv.bg;

[†]These authors contributed equally to this work.

Abstract

We generalize an abstract variational principle in Banach spaces, introduced by Topalova & Zlateva [1], by showing that the set $\mathbb{P}_0$ of perturbations for which a perturbed lower semi-continuous function f is WPMC (well posed modulus compact) not only contains a dense G_δ subset, but is also a complement to a σ -porous subset in a specifically defined positive cone. If the domain of f is bounded, then the cone is replaced by the entire perturbation space $\mathbb{P}$. Moreover, if the space is a Musielak-Orlicz sequence space satisfying $\ell_\Phi \cong h_\Phi$, then the notion WPMC is replaced by the stronger notion of Tikhonov well posedness, which is proved to be equivalent to the single-valuedness and upper semi-continuity of the multivalued mapping assigning a parameter to the solution set.

We give several applications. The first one is that the Musielak-Orlicz sequence spaces have the Radon-Nikodym property and, therefore, are dentable by proving the validity of Stegall's variational principle. As a consequence we obtain that the duals of Musielak-Orlicz sequence spaces are w^* -Asplund. We establish also a sufficient condition for Musielak-Orlicz and Nakano sequence spaces to be Asplund spaces. The next applications are for determining the type of the smoothness of certain Musielak-Orlicz, Nakano, and weighted Orlicz sequence

spaces. We illustrate by an example that it is possible to consider an Orlicz function without the Δ_2 condition, by a particular choice of the weighted sequence $\{w_n\}_{n=1}^\infty$ to get $\ell_M(w) \cong h_M(w)$ and to be able to apply the main result.

Keywords: Musielak-Orlicz sequence space, weighted Orlicz sequence spaces, Nakano sequence space, perturbation space, variational principle

MSC Classification: 46N10 , 35A15 , 49J45 , 90C48

1 Introduction

A new perturbation tool or variational principle in Orlicz sequence spaces is established Topalova & Zlateva in [1] (see Theorem 2). As an application, the authors found the best possible order of the remainder of a differentiable bump function in such spaces.

The variational principle in [1] is extended here (by adding a stronger condition) in three directions: 1) the dense G_δ subset can be replaced by a complement of σ -porous set, 2) the cone of perturbations can be replaced with the entire space and 3) if the space is a Musielak-Orlicz sequence space satisfying $\ell_\Phi \cong h_\Phi$, then the notion well-posed modulus compact (WPMC) minimization problem is replaced by the stronger Tikhonov well posedness which is proved to be equivalent to the single-valuedness and upper semi-continuity of the multivalued mapping assigning a parameter to the solution set.

As an application, we prove the Stegall variational principle in Musielak-Orlicz sequence spaces, namely, showing that the set of linear perturbations determining Tichonov well posedness contains a dense G_δ subset. As a consequence we get that Musielak-Orlicz sequence spaces have Radon-Nikodym property and, therefore, are dentable and their dual spaces are w^* -Asplund spaces. With a different argument we establish a sufficient condition for Musielak-Orlicz and Nakano sequence spaces to be Asplund spaces.

Variational principles are widely used in getting of negative results in the renorming theory of Banach spaces [2]. With the help of Ekeland variational principle upper estimates for the order differentiability of norms are obtained in ℓ_p spaces [2]. With the Stegall variational principle lower and upper estimates are obtained about for the order differentiability of norms and bump functions in certain Banach spaces [3]. We give few examples how the method can be used for determining the type of the smoothness of certain Musielak-Orlicz, Nakano, and weighted Orlicz sequence spaces. We illustrate by an example that it is possible to consider an Orlicz function without the Δ_2 condition, but by a particular choice of the weighted sequence $\{w_n\}_{n=1}^\infty$ to get $\ell_M(w) \cong h_M(w)$ and to be able to apply the main result and its corollaries.

The article is organized as follows.

In Section 2 we present the well known notions, notations, and results about abstract perturbation technique, porosity, Musielak-Orlicz sequence space, monotone mappings.

In section 3 we deliver some auxiliary results, define a perturbation space and prove that it satisfies the stronger condition in our abstract variational principle. We present a new result about monotone mapping from [4], which is the base for the proof of the main result.

In section 4 we prove the main result: a variational principle in Musielak-Orlicz sequence spaces in two main theorems.

In section 5 we apply the main result for proving the validity of Stegall's variational principle in Musielak-Orlicz sequence spaces, and therefore, such spaces possess the Radon-Nikodym property and their dual spaces are w^* -Asplund. We present applications to differentiability of bump functions in Musielak-Orlicz sequence spaces, determine the type of the smoothness in Musielak-Orlicz, Nakano, and weighted Orlicz sequence spaces.

In section 6 we give examples of weighted Orlicz spaces without Δ_2 -condition for which the main results are applicable.

2 Preliminaries

We use the standard Banach space terminology from [5]. Let $(X, \|\cdot\|)$ be a normed space and (X, ρ) be a metric spaces. Whenever we consider a metric ρ in a normed space $(X, \|\cdot\|)$ we assume $\rho(x, y) = \|x - y\|$. We denote by S_X be the unit sphere of X , and B_X be the closed unit ball of X . We will denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of the natural and real numbers, respectively. For $K > 0$ we will denote $KB_{(X, \|\cdot\|)} = \{x \in X : \|x\| \leq K\}$.

2.1 Abstract Perturbation Technique and Porosity

The main technique of [6] was refined in [1], in order to be applied in the context of Orlicz sequence spaces.

Let $(X, \|\cdot\|)$ be a Banach space and let S be a nonempty subset in X . By $\alpha(S)$ the Kuratowski index of non-compactness of S is denoted (see [7]), that is the infimum of all $\varepsilon > 0$ for which S admits finite (or compact) ε net.

For a function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\varepsilon \geq 0$ we will use the notation

$$\Omega_f^S(\varepsilon) = \left\{x \in S : f(x) \leq \inf_S f + \varepsilon\right\}.$$

Clearly, $\Omega_f^S(0) = \operatorname{argmin}_S f$.

Proposition 1 ([1]) *For any $f, g : X \rightarrow \mathbb{R} \cup \{+\infty\}$, bounded below on S , and $\delta > 0$ if $\Omega_f^S(\delta) \cap \Omega_g^S(\delta) \neq \emptyset$, then $\Omega_{f+g}^S(\delta) \subset \left(\Omega_f^S(3\delta) \cap \Omega_g^S(3\delta)\right)$.*

Definition 1 ([8]) Let $S \subset X$ and $f : S \rightarrow \mathbb{R} \cup \{+\infty\}$. We say that the minimization problem $(f, S) = \min\{f(x) : x \in S\}$ is well-posed modulus compact, abbreviated as *WPMC*, if $\inf_S f \in \mathbb{R}$ and $\Omega_f^S(0)$ is nonempty and compact set such that if $\{x_n\}_{n=1}^\infty$ is a minimizing sequence: $\lim_{n \rightarrow \infty} f(x_n) = \min_S f$, then $\lim_{n \rightarrow \infty} d(x_n, \Omega_f^S(0)) = 0$.

The meaning of the term *WPMC* is quite natural. If $\Omega_f^S(0)$ is a singleton, then we get the well-known notion of Tikhonov well-posedness (see e.g. [9], Chapter I). From a lemma of Kuratowski it follows [8] that for a closed set $S \subset X$ and a proper lower semi-continuous function $f : S \rightarrow \mathbb{R} \cup \{+\infty\}$ bounded below on S there holds

$$(f, S) \text{ is } WPMC \text{ if and only if } \lim_{t \downarrow 0} \alpha(\Omega_f^S(t)) = 0.$$

Definition 2 ([1]) Let $(X, \|\cdot\|)$ be a Banach space and $S \subseteq X$ be a non-empty set. The space $(\mathbb{P}, \|\cdot\|_{\mathbb{P}})$ of real continuous, and bounded on S functions, is called a perturbation space on S if

- (d2.i) $\mathbb{P}$ is a convex cone
- (d2.ii) $\mathbb{P}$ is complete with respect to the norm $\|\cdot\|_{\mathbb{P}}$, defined on $\text{span}\{\mathbb{P}\}$, that dominates the uniform convergence on S , i.e., for some constant $C > 0$ there holds

$$\sup_{x \in S} |f(x)| \leq C \|f\|_{\mathbb{P}}$$

for all $f \in \text{span}\{\mathbb{P}\}$ (in other words, $(\text{span}\{\mathbb{P}\}, \|\cdot\|_{\mathbb{P}})$ is a Banach space of bounded on S continuous, and $\mathbb{P}$ is its positive cone)

- (d2.iii) for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ so that for any $x \in S$ there exists $f \in \mathbb{P}$ (depending on x), so that

$$\|f\|_{\mathbb{P}} \leq \varepsilon, \quad x \in \Omega_f^S(\delta), \quad \text{and} \quad \alpha(\Omega_f^S(3\delta)) \leq \varepsilon.$$

Then, we say that $(\mathbb{P}, \|\cdot\|_{\mathbb{P}})$ is a perturbation space on S .

Theorem 2 ([1]) Let $(X, \|\cdot\|)$ be a Banach space, $S \subseteq X$ be nonempty and closed, $(\mathbb{P}, \|\cdot\|_{\mathbb{P}})$ be a perturbation space on S , and $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous and bounded below on S function.

Then for every $\varepsilon > 0$ there is $g \in \mathbb{P}$ so that $\|g\|_{\mathbb{P}} < \varepsilon$ and $(f + g, S)$ is *WPMC*. In particular, $f + g$ attains its minimum on S .

We will modify the above theorem in the following way, involving porosity notions.

Let (X, ρ) be a metric space. An open (resp. closed) ball in X with center x and radius $r > 0$ is denoted by $B(x, r)$ (resp. $\bar{B}(x, r)$). If $(X, \|\cdot\|)$ is a normed space, then $B_X := \bar{B}(0, 1)$ (resp. $S_X := \{x \in X : \|x\| = 1\}$).

Definition 3 ([10]) A subset A of a metric space (X, ρ) is said to be porous in X , if there exist $\alpha > 0$ and $r_0 > 0$ such that for every $r \in (0, r_0)$ and $x \in A$ there is $z \in X$ such that $B(z, \alpha r) \subset B(x, r) \setminus A$.

Definition 4 ([10]) A set A is called σ -porous in (X, ρ) , if it is a countable union of sets, which are porous in X .

Clearly, a set which is σ -porous in X is of first category, the converse being false in general.

Given a normed space E , for $v \in E$ and $c > 0$ define the cone

$$C(v, c) = \{x \in E : x = \lambda v + w, \lambda > 0, \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(v, c).$$

Definition 5 ([4]) A subset M of a normed space $(X, \|\cdot\|)$ is called cone porous (resp. fully cone porous) at $x \in M$, if there exist $r > 0, v \in X, \|v\| = 1$ and $c \in (0, 1)$ such that

$$M \cap B(x, r) \cap (x + C(v, c)) = \emptyset,$$

(resp. $M \cap (x + C(v, c)) = \emptyset$).

Definition 6 ([4]) The set M is called cone porous (resp. fully cone porous), if it is cone porous (resp. fully cone porous) at every of its point.

Definition 7 ([4]) M is called σ -cone porous (resp. σ -fully cone porous), if it is a union of countably many cone porous (resp. σ -fully cone porous) sets.

It is clear that every fully cone porous set is cone porous and every cone porous set is porous.

2.2 Musielak-Orlicz Sequence Spaces

Let ℓ^0 stand for the space of all real sequences, i.e., $x = \{x_i\}_{i=1}^{\infty} \in \ell^0$.

Definition 8 ([11]) A Banach space $(X, \|\cdot\|)$ is said to be Köthe sequence space if X is a subspace of ℓ^0 such that

- (d8.i) if $x \in \ell^0, y \in X$ and $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$ then $x \in X$ and $\|x\| \leq \|y\|$
- (d8.ii) there exists an element $x \in X$ such that $x_i > 0$ for all $i \in \mathbb{N}$.

Following [5], a sequence $\{v_i\}_{i=1}^{\infty}$ in a Banach space X is called Schauder basis of X (or basis for short) if for each $x \in X$ there exists a unique sequence $\{a_i\}_{i=1}^{\infty}$ of scalars such that $x = \sum_{i=1}^{\infty} a_i v_i$ and we will use the notation $\text{supp}\{x\} = \{i \in \mathbb{N} : a_i \neq 0\}$. If $\{v_i\}_{i=1}^{\infty}$ is a basis in X such that the series $\sum_{i=1}^{\infty} a_i v_i$ converges whenever

$\sup_{n \in \mathbb{N}} \left\| \sum_{i=1}^n a_i v_i \right\| < \infty$, then it is called a boundedly complete basis of X . A sequence of

non zero vectors $\{x^{(n)}\}_{n=1}^{\infty}$ of the form $\sum_{i=p_n+1}^{p_{n+1}} a_i v_i$, with $\{a_i\}_{i=1}^{\infty}$ scalars and $0 = p_1 < p_2 < p_3 \dots$ an increasing sequence of integers is called a block basic sequence or block basis of $\{v_i\}_{i=1}^{\infty}$ for short. By $\{e_i\}_{i=1}^{\infty}$ we denote the unit vectors.

Let us recall that an Orlicz function M is an even, continuous, nondecreasing convex function such that $M(0) = 0$. We say that M is non-degenerate Orlicz function if $M(t) > 0$ for every $t > 0$. A sequence $\Phi = \{\Phi_i\}_{i=1}^\infty$ of Orlicz functions is called a Musielak–Orlicz function. We say that a Musielak–Orlicz function Φ is non-degenerate one if each Φ_n are non-degenerate Orlicz functions.

The Musielak–Orlicz sequence space ℓ_Φ , generated by a Musielak–Orlicz function Φ is the set of all real sequences $\{x_i\}_{i=1}^\infty$ such that $\sum_{i=1}^\infty \Phi_i(\lambda x_i) < \infty$ for some $\lambda > 0$. The space ℓ_Φ is a Banach space if endowed with the Luxemburg’s norm:

$$\|x\|_\Phi = \inf \left\{ r > 0 : \sum_{i=1}^\infty \Phi_i(x_i/r) \leq 1 \right\}.$$

To make it easier to write in some of the formulas, we will use $\sum_{k \in \text{supp}\{x\}} \Phi_k(x_k)$ for $x = \{x_i\}_{i=1}^\infty \in \ell_\Phi$.

Throughout this note we always denote by M an Orlicz function and by Φ a Musielak–Orlicz function.

If the Musielak–Orlicz function Φ consists of one and the same Orlicz function M we get the Orlicz sequence space denoted by ℓ_M .

A weight sequence $w = \{w_i\}_{i=1}^\infty$ is a sequence of positive reals. We will distinguish two classes of weighted sequences Λ_∞ and Λ . The weight sequence $w = \{w_i\}_{i=1}^\infty$ is from the class Λ_∞ if it is nondecreasing sequence with $\lim_{i \rightarrow \infty} w_i = \infty$. Following [12, 13] we say that $w = \{w_i\}_{i=1}^\infty$ is from the class Λ if there exists a subsequence $w = \{w_{i_k}\}_{k=1}^\infty$ such that $\lim_{k \rightarrow \infty} w_{i_k} = 0$ and $\sum_{k=1}^\infty w_{i_k} = \infty$.

A weighted Orlicz sequence space $\ell_M(w)$ generated by an Orlicz function M and a weight sequence w is the Musielak–Orlicz sequence space ℓ_Φ , where $\Phi_i(t) = w_i M(t)$.

Weighted Orlicz sequence spaces were investigated for example in [13–16]. Let us mention that if the weight sequence is from the class Λ , then a lot of the properties of the space $\ell_M(w)$ depend only on the generating Orlicz function M , which is in contrast with the results when $w \notin \Lambda$ [12, 17, 18].

It is well known that the Orlicz and the weighted Orlicz sequence spaces equipped with Luxemburg norm are Köthe sequence spaces. For simplicity of notations we will use $\tilde{\Phi}(x) = \sum_{i=1}^\infty \Phi_i(x_i)$ and $\tilde{M}_w(x) = \sum_{i=1}^\infty w_i M(x_i)$.

An extensive study of Orlicz and Musielak–Orlicz spaces can be found in [5, 19].

We denote by h_Φ the closed linear subspace of ℓ_Φ , generated by all $x \in \ell_\Phi$, such that $\tilde{\Phi}(\lambda x_i) < \infty$ for every $\lambda > 0$ and by $h_M(w)$ the subspace of $\ell_M(w)$ such that $\tilde{M}_w(\lambda x) < \infty$ for every $\lambda > 0$.

The unit vectors $\{e_i\}_{i=1}^\infty$ form a boundedly complete basis in h_Φ , equipped with the Luxemburg norm.

We say that M has Δ_2 -condition if there exist $C > 1$ and $t_0 > 0$ such that $M(2t) \leq CM(t)$ for every $t \in (0, t_0]$.

If $w \in \Lambda$ then the spaces $\ell_M(w)$ and $h_M(w)$ coincide if and only if $M \in \Delta_2$. The proof is similar to that done in ([5] Proposition 4.a.4).

For a given Orlicz function M the following numbers are associated (see [5], p. 143):

$$\alpha_M = \sup\{p : \sup\{M(uv)/u^p M(v) : u, v \in (0, 1]\} > 0\},$$

$$\beta_M = \inf\{p : \inf\{M(uv)/u^p M(v) : u, v \in (0, 1]\} > 0\}.$$

An Orlicz function M satisfies the Δ_2 -condition if and only if $\beta_M < \infty$, which implies of course $M(uv) \geq u^q M(v)$, $u, v \in [0, 1]$ for some $q \geq \beta_M$ (see [5] p.140).

Definition 9 ([20]) We say that the Musielak–Orlicz function Φ satisfies the δ_2 -condition if there exist constants $K, \beta > 0$ and a non-negative sequence $\{c_n\}_{n=1}^\infty \in \ell_1$ such that for every $n \in \mathbb{N}$

$$\Phi_n(2t) \leq K\Phi_n(t) + c_n, \quad (1)$$

provided $t \in [0, \Phi_n^{-1}(\beta)]$.

The spaces ℓ_Φ and h_Φ coincide if and only if Φ has δ_2 -condition.

We say that the Musielak–Orlicz function Φ satisfies the uniform δ_2 -condition if it satisfies (1) for every $t \in [0, t_0]$ for some $t_0 > 0$ with $c_n = 0$ for every $n \in \mathbb{N}$.

Recall that for given Musielak–Orlicz functions Φ and Ψ the spaces ℓ_Φ and ℓ_Ψ coincide with equivalence of norms if and only if Φ is equivalent to Ψ at 0, that is there exist constants $K, \beta > 0$ and a non-negative sequence $\{c_n\}_{n=1}^\infty \in \ell_1$, such that for every $n \in \mathbb{N}$ the inequalities

$$\Phi_n(Kt) \leq \Psi_n(t) + c_n \quad \text{and} \quad \Psi_n(Kt) \leq \Phi_n(t) + c_n$$

hold for every $t \in [0, \min(\Phi_n^{-1}(\beta), \Psi_n^{-1}(\beta))]$, [20, 21].

If $\Phi_i(1) = 1$ for every $i \in \mathbb{N}$, then the unit vectors $\{e_i\}_{i=1}^\infty$ is a normalized boundedly complete basis in h_Φ . If $\Phi_i(1) \neq 1$, and a_i is the solution of the equation $\Phi_i(a_i) = 1$, $i \in \mathbb{N}$, then ℓ_Φ , equipped with the Luxemburg norm, is isometric to ℓ_ϕ , where the Musielak–Orlicz function ϕ is defined by the sequence $\phi_i(t) = \frac{\Phi_i(a_i t)}{\Phi_i(a_i)}$. Therefore the sequence $v_i = a_i e_i$, $i \in \mathbb{N}$ is a normalized boundedly complete basis in h_Φ . The unit vector basis $\{e_i\}_{i=1}^\infty$ is a boundedly complete basis in ℓ_ϕ if and only if $\ell_\phi \cong h_\phi$. The weighted Orlicz sequence space $\ell_M(w)$ is isometric to ℓ_ϕ , where the Musielak–Orlicz function ϕ is defined by

$$\phi_i(t) = \frac{M(a_i t)}{M(a_i)}, \quad a_i = M^{-1}(1/w_i).$$

Hence $\ell_M(w) \cong h_M(w)$ if and only if $\ell_\phi \cong h_\phi$ and $y = \sum_{i=1}^\infty x_i e_i \in h_\phi$ if and only if $x = \sum_{i=1}^\infty a_i x_i e_i \in h_M(w)$. The spaces $\ell_M(w)$ and ℓ_ϕ are isometric.

For any Orlicz function M the function $M^*(t) = \sup_{s \geq 0} \{st - M(s)\}$ is called a conjugate function to M . It plays a crucial role in the investigation of the dual space ℓ_M^* . For a Musielak–Orlicz function $\Phi = \{\Phi_n\}_{n=1}^\infty$ the conjugated function is $\Phi^* = \{\Phi_n^*\}_{n=1}^\infty$. It is well known that if $\Phi \in \delta_2$, then ℓ_Φ^* is isomorphic to ℓ_{Φ^*} . Moreover ℓ_Φ is reflexive if $\Phi, \Phi^* \in \delta_2$ [22].

2.3 Fréchet Differentiability in Musielak-Orlicz Sequence Spaces

The existence of Fréchet smooth norms and bump functions and its impact on the geometrical properties of a Banach space have been subject to many investigations [2]. It is well known that the usual norm in ℓ_p and L_p is $p - 1$ times Fréchet differentiable for p an odd number, $[p]$ times Fréchet differentiable for $p \notin \mathbb{N}$, and infinitely many times Fréchet differentiable for p an even number [23, 24].

An extensive study and bibliography may be found in [2].

The exact order of differentiability in Orlicz spaces have been found in [25–27].

We will investigate only Fréchet differentiable norms and bump functions, and for the sake of space we will refer sometimes it just as differentiable or smooth ones.

As a Fréchet smooth norm immediately produces a bump function of the same smoothness, all negative results about bump functions are negative results about the smoothness in the class of all equivalent norms.

Let X be a Banach space and let $f : X \rightarrow \mathbb{R}$. The function f is called a smooth one at $x \in X$, if the following representation holds:

$$f(x + y) = f(x) + f'(x)(y) + R(x, y)$$

for every $y \in X$, where $f'(x)$ is a linear bounded symmetric forms on X and

$$\lim_{\|y\| \rightarrow 0} \frac{|R(x, y)|}{\|y\|} = 0.$$

A norm $\|\cdot\|$ is referred as a Fréchet differentiable if it is smooth at all points $x \in S_X$, the unit sphere of X .

Sometimes it is interesting and applicable to know the best possible order of the remainder $R(x, y)$ and thus we get the next notion. We shall say that f is F_ω -smooth at $x \in X$ for some $\omega : (0, 1] \rightarrow \mathbb{R}^+$, $\lim_{t \rightarrow 0} \frac{\omega(t)}{t^p} = 0$, $p \in [1, 2]$, if

$$\lim_{\|y\| \rightarrow 0} \frac{|R(x, y)|}{\omega(\|y\|)} = 0.$$

Let $\Phi = \{\Phi_n\}_{n=1}^\infty$ be a Musielak–Orlicz function. Let us define by

$$\psi_n(t) = \int_0^t \frac{\Phi_n(u)}{u} du$$

and

$$\Psi_n(t) = \int_0^t \frac{\psi_n(u)}{u} \exp \frac{u}{u-t} du.$$

Following [21], to every Musielak–Orlicz function $\Phi = \{\phi_n\}_{n=1}^\infty$ we associate the numbers

$$\alpha_\Phi = \liminf_{n \rightarrow \infty} \alpha_{\Phi_n} \text{ and } \beta_\Phi = \limsup_{n \rightarrow \infty} \beta_{\Phi_n}.$$

Lemma 3 ([21]) Let $\Phi = \{\Phi_n\}_{n=1}^\infty$ be a Musielak-Orlicz function. Then

1. Φ_n is equivalent to Ψ_n at 0
2. $t^k |\Psi_n^{(k)}(t)| \leq c_k \Psi_n(c_k t)$, $t \in [0, \infty)$, $c_k > 0$, $k = 1, 2, \dots$
3. ℓ_Ψ is isomorphic to ℓ_Φ
4. $\alpha_\Psi = \alpha_\Phi$.

for every n .

$$\text{For } p \geq 1 \text{ define } E(p) = \begin{cases} p - 1, & p \in \mathbb{N}, \\ [p], & p \notin \mathbb{N}. \end{cases}$$

Theorem 4 ([21]) Let Ψ be a Musielak-Orlicz function and $1 \leq k = E(\alpha_\Psi)$. Then there exists an equivalent k -times Fréchet differentiable norm in h_Ψ . If in addition $\beta_\Phi < +\infty$ then there exists an equivalent k -times uniformly Fréchet differentiable norm in ℓ_Φ .

Following ([2], p. 58), a direct consequence of Theorem 4 is the next corollary.

Corollary 5 Let Ψ be a Musielak-Orlicz function and $1 < \alpha_\Psi$, then ℓ_Ψ is an Asplund space.

2.4 Monotone Mappings

Let X and Y be Banach spaces such that $Y = X^*$ or $X = Y^*$. We remind the following notion.

Definition 10 A multivalued mapping $F : X \rightarrow 2^Y$ is called monotone if

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \quad \forall x_i \in X, y_i \in F(x_i), i = 1, 2, \quad (2)$$

where $\langle \cdot, \cdot \rangle$ are the duality brackets between X and Y .

Example 1 ([4]) The multivalued mapping $F(x) = \operatorname{argmax}_{y \in S} \{\langle x, y \rangle + f(y)\}$, where $f : Y \rightarrow \mathbb{R}$ is any function, $S \subset Y$ is any subset, is monotone.

We will need the following theorem from [4].

Theorem 6 ([4], Theorem 2.2). Suppose X and Y are Banach spaces such that $Y = X^*$ or $X = Y^*$, Y is separable, and $T : X \rightarrow 2^Y$ is a locally bounded monotone operator with domain $\operatorname{Dom}(T) = \{x \in X : T(x) \neq \emptyset\} \neq \emptyset$. Then there exists a σ -fully cone porous set $X_0 \subset \operatorname{Dom}(T)$ such that T is single-valued and upper semi-continuous on $\operatorname{Dom}(T) \setminus X_0$.

Let $f : S \rightarrow \mathbb{R}$ be a lower semi-continuous, proper and bounded from below function and $\Phi = \{\Phi_n\}_{n=1}^\infty$ be a non-degenerate Musielak-Orlicz function. For $x \in h_\Phi$

and $S \subset h_\Phi$ let us introduce the notations

$$\tilde{x} = (\Phi_1(x_1), \dots, \Phi_n(x_n), \dots) \in \ell_1,$$

$$\tilde{S} = \{\tilde{x} \in \ell_1 : x \in S\},$$

and the function $\tilde{f} : \tilde{S} \rightarrow \mathbb{R}$ be defined by $\tilde{f}(\tilde{x}) = f(x)$. Define the multivalued mappings $F : \ell_\infty \rightarrow 2^S$ and $\tilde{F} : \ell_\infty \rightarrow 2^{\tilde{S}}$ respectively, as

$$F(a) = \operatorname{argmin}_{x \in S} \left\{ \sum_{i=1}^{\infty} a_i \Phi_i(x_i) + f(x) \right\}. \quad (3)$$

and

$$\tilde{F}(a) = \operatorname{argmax}_{\tilde{x} \in \tilde{S}} \left\{ \langle -a, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \right\} \quad (4)$$

We have $x \in F(a)$ if and only if $\tilde{x} \in \tilde{F}(a)$. Also, $\tilde{F}$ is monotone w.r.t. the duality brackets $\langle \cdot, \cdot \rangle$ according to the above example, so it is monotone in the usual sense from convex analysis.

3 Auxiliary Results

To ease the proofs of the main results, we will present some intermediate ones in this section. Some of these results have their own value and can be applied in subsequent research, and this is another reason to formulate them separately instead of appearing as steps in the proofs of the main theorems.

3.1 Auxiliary Results About σ -Porous Set

We will need the next definition which is a generalization of Definition 2 introduced in [1].

Definition 11 We will say that the space $(\mathbb{P}, \|\cdot\|_{\mathbb{P}})$ of real continuous, and bounded on S functions, is a *pertrubation space in a strong sense* on S if the conditions (d2.i), (d2.ii), and (dS.iii) for any $n \in \mathbb{N}$ there exists $K_n \in (0, 6C)$ so that for any $\varepsilon > 0$, for any $x \in S$ there exists $g \in \mathbb{P}$ (depending on x and ε), such that for $\delta = K_n \varepsilon$,

$$\|g\|_{\mathbb{P}} \leq \varepsilon, \quad x \in \Omega_g^S(\delta), \quad \text{and} \quad \alpha(\Omega_g^S(3\delta)) \leq \frac{1}{n} \quad (5)$$

are satisfied.

Theorem 7 Let $(X, \|\cdot\|)$ be a Banach space, $S \subseteq X$ be nonempty and closed, $(\mathbb{P}, \|\cdot\|_{\mathbb{P}})$ be a pertrubation space on S in the strong sense. Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous and bounded below on S function. Then the set

$$A = \left\{ h \in \mathbb{P} : \text{the minimization problem } (f + h, S) \text{ is not WPMC} \right\}$$

is a σ -porous set.

Proof Define A_n by

$$A_n := \left\{ h \in \mathbb{P} : \exists t > 0 : \alpha(\Omega_{f+h}^S(t)) < \frac{1}{n} \right\}. \quad (6)$$

Let $h \in \mathbb{P}$ and $\varepsilon > 0$ be given, and K_n be provided by (dS.iii) of Definition 11.

Fix $x \in \Omega_{f+h}^S(\delta)$, and let $g \in \mathbb{P}$ be produced by the condition (dS.iii), so g satisfies (5), in particular, $x \in \Omega_g^S(\delta)$.

Claim: For any proper and bounded below function $\phi : S \rightarrow \mathbb{R}$ and any function $u : S \rightarrow \mathbb{R}$ with $\|u\|_\infty < \delta/3$, we have

$$\Omega_{\phi+u}^S(\delta/3) \subset \Omega_\phi^S(\delta).$$

Indeed, let $x \in \Omega_{\phi+u}^S(\delta/3)$. Then

$$\phi(x) < \phi(x) + u(x) + \delta/3 \leq \inf_S(\phi + u) + 2\delta/3 < \inf_S \phi + \delta.$$

and the Claim is proved.

Take $u \in \mathbb{P} : \|u\| < \delta/3C$. It follows from (d2.ii) that

$$\|u\|_\infty < \delta/3.$$

We apply the Claim for $\Phi = f + g + h$ and obtain:

$$\begin{aligned} \Omega_{f+h+g+u}^S(\delta/3) &\subset \Omega_{f+h+g}^S(\delta) \\ &\subset \Omega_g^S(3\delta) \quad (\text{by Proposition 1}). \end{aligned}$$

Now (6) implies

$$\alpha\left(\Omega_{f+h+g+u}^S(\delta/3)\right) < 1/n.$$

This means $h + g + u \in A_n$. Since this is valid for any $u \in \mathbb{P}$ with $\|u\| < \delta/3C$, we have

$$h + g + \mathbb{P} \cap B(0, \delta/3C) \subset A_n. \quad (7)$$

Take $x_0 \in \text{int}\mathbb{P}$ with $\|x_0\| = 1$. There exists $r_0 \in (0, 1)$ such that $B(x_0, r_0) \subset \text{int}\mathbb{P}$. Since

$$\frac{\delta}{6C}x_0 + r_0 \frac{\delta}{6C}B(0, 1) \subset \frac{\delta}{3C}B(0, 1)$$

and $\delta/3C < 2\varepsilon$, we have

$$B\left(h + g + \frac{\delta}{6C}x_0, r_0 \frac{\delta}{6C}\right) \subset B(h, 3\varepsilon) \quad (8)$$

Also,

$$B\left(h + g + \frac{\delta}{6C}x_0, r_0 \frac{\delta}{6C}\right) = h + g + B\left(\frac{\delta}{6C}x_0, \frac{r_0\delta}{6C}\right) \quad (9)$$

$$= h + g + \frac{\delta}{6C}B(x_0, r_0) \quad (10)$$

$$\subset h + g + \mathbb{P} \cap B\left(0, \frac{\delta}{3C}\right) \quad (11)$$

$$\subset A_n \quad \text{by (7)}$$

This implies that the set $P \setminus A_n$ is porous, because of (8) and because $\frac{r_0\delta/6C}{3\varepsilon} = \frac{r_0K_n}{18C}$ is a constant not depending on h . By the Baire category theorem the set

$$\mathbb{P} \setminus \mathbb{P}_0 \quad \text{where} \quad \mathbb{P}_0 := \bigcap_{n=1}^{\infty} A_n \neq \emptyset,$$

is σ -porous. □

3.2 Auxiliary Results About Musielak-Orlicz Sequence Spaces

Lemma 8 *Let Φ be a non-degenerate Musielak-Orlicz function. If there is a sequence $\{x^{(n)}\}_{n=1}^{\infty} \subset \ell_{\Phi}$, so that $\inf \{\|x^{(n)}\|_{\Phi} : n \in \mathbb{N}\} > 0$ and the equality*

$$\lim_{n \rightarrow \infty} \tilde{\Phi}(x^{(n)}) = \sum_{i=1}^{\infty} \Phi_i(x_i^{(n)}) = 0 \quad (12)$$

holds, then $h_{\Phi} \not\subseteq \ell_{\Phi}$, i.e., there exists $x \in \ell_{\Phi}$ and $x \notin h_{\Phi}$.

Proof Let us put $I = \inf \{\|x^{(n)}\|_{\Phi} : n \in \mathbb{N}\}$. Then for every $n \in \mathbb{N}$ there holds

$$\tilde{\Phi}\left(\frac{x^{(n)}}{I}\right) = \sum_{i=1}^{\infty} \Phi_i\left(\frac{x_i^{(n)}}{I}\right) \geq 1. \quad (13)$$

There are two cases: I) for every $i \in \mathbb{N}$ there holds $\lim_{n \rightarrow \infty} \Phi_i\left(\frac{x_i^{(n)}}{I}\right) = 0$ and II) case if I) does not hold.

Let us assume first that there holds case I), i.e., for every $i \in \mathbb{N}$ there holds

$$\lim_{n \rightarrow \infty} \Phi_i\left(\frac{x_i^{(n)}}{I}\right) = 0.$$

We observe that for every $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$, such that if $n \geq N$ then

$$\tilde{\Phi}\left(\frac{\sum_{i=1}^m x_i^{(n)} e_i}{I}\right) \leq \frac{1}{4}. \quad (14)$$

By the definition of I and the assumption $\ell_{\Phi} \cong h_{\Phi}$, it follows that for every $n \in \mathbb{N}$, there holds

$$1 \leq \tilde{\Phi}\left(\frac{x^{(n)}}{I}\right) = \sum_{i=1}^{\infty} \Phi_i\left(\frac{x_i^{(n)}}{I}\right) < \infty. \quad (15)$$

From (15), for every $n \in \mathbb{N}$, there is $p_n \in \mathbb{N}$ such that

$$\sum_{i=p_n+1}^{\infty} \Phi_i\left(\frac{x_i^{(n)}}{I}\right) = \tilde{\Phi}\left(\frac{\sum_{i=p_n+1}^{\infty} x_i^{(n)} e_i}{I}\right) \leq \frac{1}{4}. \quad (16)$$

From (14), (15), and (16), we get that for every $N_1, N_2 \in \mathbb{N}$, there is $n \geq N_1$ such that

$$\tilde{\Phi}\left(\frac{\sum_{i=N_2}^{p_n} x_i^{(n)} e_i}{I}\right) \geq \frac{1}{2}. \quad (17)$$

By (12), (13), and (17), we observe that for every $n, N \in \mathbb{N}$ there exists $j_{N,n} \in \mathbb{N}$, such that

$$\tilde{\Phi}\left(\frac{\sum_{i=N}^{p_{j_{N,n},n}} x_i^{(j_{N,n},n)} e_i}{I}\right) \geq \frac{1}{2} \quad \text{and} \quad \tilde{\Phi}\left(\sum_{i=N}^{p_{j_{N,n},n}} x_i^{(j_{N,n},n)} e_i\right) \leq \frac{1}{2^n}. \quad (18)$$

Let us define the sequence $\{q_n\}_{n=1}^{\infty}$ as $q_1 = 0$ and $q_{n+1} = p_{j_{q_n+1,n}}$. There holds $q_1 < q_2 < q_3 \dots$. Let us consider $x^* = \sum_{i=1}^{\infty} \sum_{s=q_n+1}^{q_{n+1}} x_i^{(j_{q_n+1,n})} e_i$. Using (18), it follows that

$$\tilde{\Phi}(x^*) = \sum_{n=1}^{\infty} \tilde{\Phi}\left(\sum_{i=q_n+1}^{q_{n+1}} x_i^{(j_{q_n+1,n})} e_i\right) = \sum_{n=1}^{\infty} \tilde{\Phi}\left(\sum_{i=q_n+1}^{p_{j_{q_n+1,n}}} x_i^{(j_{q_n+1,n})} e_i\right) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \leq 1$$

and

$$\begin{aligned}\tilde{\Phi}\left(\frac{x^*}{I}\right) &= \sum_{n=1}^{\infty} \tilde{\Phi}\left(\frac{1}{I} \sum_{i=q_n+1}^{q_{n+1}} x_i^{(j_{q_n+1,n})} e_i\right) = \sum_{n=1}^{\infty} \tilde{\Phi}\left(\frac{1}{I} \sum_{i=q_n+1}^{p_{jq_n,n}} x_i^{(j_{q_n+1,n})} e_i\right) \\ &\geq \sum_{n=1}^{\infty} \frac{1}{2} = +\infty,\end{aligned}$$

which contradicts with the assumption $\ell_{\Phi} \cong h_{\Phi}$.

If case I) does not hold, there is $i_0 \in \mathbb{N}$ such that $\limsup_{n \rightarrow \infty} \Phi_{i_0}\left(\frac{x^{(n)}}{I}\right) > 0$. Thus there is a subsequence $\{x^{(n_k)}\}_{k=1}^{\infty}$ so that

$$\lim_{k \rightarrow \infty} \Phi_{i_0}\left(\frac{x_{i_0}^{(n_k)}}{I}\right) > 0. \quad (19)$$

Then, from (12) it follows that

$$\lim_{k \rightarrow \infty} \tilde{\Phi}\left(x^{(n_k)}\right) = \lim_{k \rightarrow \infty} \sum_{i=1}^{\infty} \Phi_i\left(x_i^{(n_k)}\right) = 0. \quad (20)$$

By (19) and (20), it follows that for the sequence $\{r_k\}_{k=0}^{\infty} \subset \mathbb{R}$, $r_k = x_{i_0}^{(n_k)}$ for $k \in \mathbb{N}$ there hold

$$\lim_{k \rightarrow \infty} \Phi_{i_0}(r_k) = 0 \quad (21)$$

and

$$\lim_{k \rightarrow \infty} \Phi_{i_0}\left(\frac{r_k}{I}\right) > 0. \quad (22)$$

Since Φ_{i_0} is a non-degenerated, Φ_{i_0} has a strong minimum at zero. Then, using (21), we get that $\lim_{k \rightarrow \infty} r_k = 0$. Hence, $\lim_{k \rightarrow \infty} \frac{r_k}{I} = 0$, but (22) contradicts with the assumption that Φ_{i_0} is continuous at 0. $\square$

Lemma 9 *Let Φ be a non-degenerate Musielak-Orlicz function. Then $h_{\Phi} \cong \ell_{\Phi}$ if and only if $\tilde{\Phi}(x)$ has a strong minimum at zero.*

Proof Let there holds $h_{\Phi} \cong \ell_{\Phi}$. For every $n, m \in \mathbb{N}$, and $x = \sum_{k=1}^{\infty} x_k e_k \in \ell_{\Phi}$ there holds

$$\tilde{\Phi}(x) = \sum_{i=1}^{\infty} \Phi_i(x_i) \geq \sum_{i=n}^m \Phi_i(x_i) = \tilde{\Phi}\left(\sum_{i=n}^m x_i e_i\right).$$

Consequently $\min_{x \in \ell_{\Phi}} \tilde{\Phi}(x) = 0$ and $\tilde{\Phi}(0) = 0$, where 0 is the zero vector in ℓ_{Φ} .

Let us suppose that $\tilde{\Phi}$ does not have a strong minimum at zero. Thus, there is $\{x^{(n)}\}_{n=1}^{\infty} \subset \ell_{\Phi}$, so that $\inf \left\{ \|x^{(n)}\|_{\Phi} : n \in \mathbb{N} \right\} > 0$ and the following equality holds:

$$\lim_{n \rightarrow \infty} \tilde{\Phi}\left(x^{(n)}\right) = \sum_{i=1}^n \Phi_i\left(x_i^{(n)}\right) = 0.$$

By Lemma 8 it follows that there is $x \in \ell_{\Phi}$ and $x \notin h_{\Phi}$, i.e., $h_{\Phi} \not\cong \ell_{\Phi}$, a contradiction.

Let now $\tilde{\Phi}$ has a strong minimum at zero.

Suppose the contrary, i.e., $h_\Phi \neq \ell_\Phi$. Hence, there is $x^* \in \ell_\Phi$, $x^* \notin h_\Phi$. Therefore there is $\lambda > 0$, such that $\tilde{\Phi}(x^*) < \infty$ and $\tilde{\Phi}\left(\frac{x^*}{\lambda}\right) = \infty$. Consequently, for every $n \in \mathbb{N}$ there is $m_n \in \mathbb{N}$, such that

$$1 \leq \tilde{\Phi}\left(\frac{\sum_{i=n+1}^{m_n} x_i^*}{\lambda}\right) < \infty. \quad (23)$$

Let us define the sequence $\{x^{(n)}\}_{n=1}^\infty \subset \ell_\Phi$ by $x^{(n)} = \sum_{i=n+1}^{m_n} x_i^* e_i$. From (23) we get

$$\inf \left\{ \left\| \frac{x^{(n)}}{\lambda} \right\|_\Phi : n \in \mathbb{N} \right\} \geq 1.$$

Consequently $\inf_{n \in \mathbb{N}} \|x_n\|_\Phi \geq \lambda$.

From $\tilde{\Phi}(x^*) < \infty$ we get $\lim_{n \rightarrow \infty} \sup_{k \geq n+1} \tilde{\Phi}\left(\sum_{i=n+1}^k x_i^* e_i\right) = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \tilde{\Phi}\left(\sum_{i=n+1}^{m_n} x_i^* e_i\right) = 0,$$

i.e., $\lim_{n \rightarrow \infty} \tilde{\Phi}(x^{(n)}) = 0$, which is a contradiction with the assumption that $\tilde{\Phi}$ has a strong minimum at zero. $\square$

Let Φ be a Musielak-Orlicz function. For every $a \in \ell_\infty$, $x \in \ell^0$ we will use the notation

$$\tilde{\Phi}_a(x) = \sum_{i=1}^\infty a_i \Phi_i(x_i).$$

If $\{a_i\}_{i=1}^\infty$ is the constant sequence $a_i = 1$ for all $i \in \mathbb{N}$, we get $\tilde{\Phi}(x) = \tilde{\Phi}_{\{1\}_{i=1}^\infty}(x)$.

Let us recall that we consider the space ℓ_∞ endowed with the supremum norm, i.e., for any $a = \{a_i\}_{i=1}^\infty \in \ell_\infty$ we put $\|a\|_\infty = \sup\{|a_i| : i \in \mathbb{N}\}$.

Let Φ be a Musielak-Orlicz function and $h_\Phi \cong \ell_\Phi$, then $\sup \left\{ \tilde{\Phi}(x) : \|x\|_\Phi \leq t \right\} < \infty$ for any $t \geq 0$. Let us denote $\nu(t) = \sup \left\{ \tilde{\Phi}(x) : \|x\|_\Phi \leq t \right\}$.

Lemma 10 *Let Φ be a not-degenerated Musielak-Orlicz function and $h_\Phi \cong \ell_\Phi$. Then for every $K > 0$ the function $\tilde{\Phi}_a$ is bounded and Lipschitz on KB_{ℓ_Φ} , i.e.,*

$$\sup_{x \in KB_{\ell_\Phi}} |\Phi_a(x)| \leq \|a\|_\infty \nu(K) \quad (24)$$

and

$$|\Phi_a(x) - \Phi_a(y)| \leq 2\|a\|_\infty \nu(K+1) \|x - y\|_\Phi$$

for every $x, y \in KB_{\ell_\Phi}$.

Proof We prove the lemma in two steps. Let us first assume that $a \in \ell_\infty^+$, i.e., $a_i \geq 0$ for all $i \in \mathbb{N}$. Then $\tilde{\Phi}_a(x) \geq 0$ and

$$\tilde{\Phi}_a(x) = \sum_{i=1}^\infty a_i \Phi_i(x_i) \leq \|a\|_\infty \sum_{i=1}^\infty \Phi_i(x_i) = \|a\|_\infty \tilde{\Phi}(x),$$

so (24) is fulfilled, i.e., $\tilde{\Phi}_a$ is bounded.

Since $\tilde{\Phi}_a$ is convex, it has directional derivatives $\tilde{\Phi}'_a(x; h)$. If $\|x\|_\Phi \leq 1$ and $\|h\|_\Phi = 1$ then by convexity $\tilde{\Phi}_a(x + h) \geq \tilde{\Phi}_a(x) + \tilde{\Phi}'_a(x; h)$. From $x + h \in (K + 1)B_{\ell_\Phi}$ it follows that $\tilde{\Phi}_a(x + h) \leq \|a\|_\infty \nu(K + 1)$ and $\tilde{\Phi}_a(x) \geq 0$, and consequently

$$\tilde{\Phi}'_a(x; h) \leq \|a\|_\infty \nu(K + 1)$$

for all $x \in KB_{\ell_\Phi}$ and all $h \in S_{\ell_M}$. Thus $\tilde{\Phi}_a(x)$ is Lipschitz on KB_{ℓ_Φ} with a Lipschitz constant $\|a\|_\infty \nu(K + 1)$.

For the second step in the proof let now for an arbitrary $a \in \ell_\infty$ we set the canonical representation $a = (a_+) - (a_-)$, where $(a_+)_n = \max\{a_n, 0\}$, so $(a_-)_n = -\min\{a_n, 0\}$. there holds $\tilde{\Phi}_a = \tilde{\Phi}_{a_+} - \tilde{\Phi}_{a_-}$ and since $a_+, a_- \in \ell_\infty^+$, $\|a\|_\infty = \max\{\|a_+\|_\infty, \|a_-\|_\infty\}$, the claim follows. $\square$

Set $\mathbb{P} = \{\tilde{\Phi}_a : a \in \ell_\infty^+\}$, which is the cone of positive bounded sequences. We will consider the $\text{span}\{\mathbb{P}\}$, i.e., the set of all functions $\tilde{\Phi}_a, a \in \ell_\infty$. Let us denote $\|\tilde{\Phi}_a\|_{\mathbb{P}} = \|a\|_\infty$.

It is easy to see that $(\text{span}\{\mathbb{P}\}, \|\cdot\|_{\mathbb{P}})$ and $(\ell_\infty, \|\cdot\|_\infty)$ are isometric.

Lemma 11 $(\text{span}\{\mathbb{P}\}, \|\cdot\|_{\mathbb{P}})$ is a perturbation space in the strong sense on each nonempty, closed and bounded subset S of ℓ_Φ , provided that $h_\Phi = \ell_\Phi$.

Proof We need to check the axioms of Definition 11, i.e. (d2.i), (d2.ii), and (dS.iii) for an arbitrary ball KB_{ℓ_Φ} . By the boundedness of S it follows that there is $K > 0$, so that $S \subset KB_{\ell_\Phi}$.

The functions from $\text{span}\{\mathbb{P}\}$ are continuous due to Lemma 10. The assumption (d2.i) is clear and (d2.ii) follows from (24).

Let us denote $\varphi(t) = \text{diam}(\Omega_{\tilde{\Phi}}(t))$. Then, by Lemma 9,

$$\lim_{t \rightarrow 0} \varphi(t) = 0 \quad (25)$$

To check (dS.iii), let $n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary chosen. From (25) there is $K_n > 0$, such that

$$\varphi(3K_n) < \frac{1}{n}. \quad (26)$$

Put $\delta = K_n \varepsilon$. There exists $\theta \in (0, \varepsilon)$ so that

$$\theta \nu(K) < \frac{\delta}{2}. \quad (27)$$

Let us chose an arbitrary $x \in S \subset KB_{\ell_\Phi}$. Hence $\|x\| \leq K$. Chose $N \in \mathbb{N}$ such that

$$\varepsilon \tilde{\Phi} \left(\sum_{i=N+1}^{\infty} x_i e_i \right) = \varepsilon \sum_{i=N+1}^{\infty} \Phi_i(x_i) < \frac{\delta}{2}, \quad (28)$$

and put

$$\bar{a} = \theta \sum_{i=1}^N e_i + \varepsilon \sum_{i=N+1}^{\infty} e_i.$$

Denote

$$g(y) := \tilde{\Phi}_{\bar{a}}(y) = \theta \tilde{\Phi} \left(\sum_{i=1}^N y_i e_i \right) + \varepsilon \tilde{\Phi} \left(\sum_{i=N+1}^{\infty} y_i e_i \right).$$

To finish the proof we need to verify that g satisfies (dS.iii), i.e., (5).

We have $\|g\|_{\mathbb{P}} = \|\bar{a}\|_{\infty} = \varepsilon$,

$$\tilde{\Phi} \left(\sum_{i=1}^N x_i e_i \right) \leq \tilde{\Phi} \left(\sum_{i=1}^{\infty} x_i e_i \right) \leq \nu(K),$$

and by (27)

$$\theta \tilde{\Phi} \left(\sum_{i=1}^N x_i e_i \right) < \frac{\delta}{2}. \quad (29)$$

Using (28) and (29) we get $\tilde{\Phi}_{\bar{a}}(x) < \delta$. Since $\tilde{\Phi}_{\bar{a}}(0) = 0$ we conclude that $\min \left\{ \tilde{\Phi}_{\bar{a}}(x) : x \in S \right\} = 0$ and thus $x \in \Omega_{\tilde{\Phi}_{\bar{a}}}^S(\delta)$.

Let $y \in \Omega_{\tilde{\Phi}_{\bar{a}}}^S(3\delta)$, i.e., $\|y\| \leq K$ and $\tilde{\Phi}_{\bar{a}}(y) \leq 3\delta$. In particular,

$$\varepsilon \tilde{\Phi} \left(\sum_{i=N+1}^{\infty} y_i e_i \right) \leq 3\delta$$

which is equivalent to $\sum_{i=N+1}^{\infty} y_i e_i \in \Omega_{\tilde{\Phi}}^S \left(\frac{3\delta}{\varepsilon} \right) = \Omega_{\tilde{\Phi}}^S(3K_n)$. From $0 \in \Omega_{\tilde{\Phi}}(3K_n)$, and (26) we conclude that $\left\| \sum_{i=N+1}^{\infty} y_i e_i \right\| < \frac{1}{n}$.

Denote $A = \left\{ \sum_{i=1}^N z_i e_i : \sum_{i=1}^{\infty} z_i e_i \in S \right\}$, which is compact.

Since $\sum_{i=1}^N y_i e_i \in A$ and $y = \sum_{i=1}^N y_i e_i + \sum_{i=N+1}^{\infty} y_i e_i$ it follows that

$$\text{dist}(y, A) < \frac{1}{n}.$$

By the arbitrary choice of $y \in \Omega_g^S(3\delta)$ we get that $\alpha \left(\Omega_g^S(3\delta) \right) \leq \frac{1}{n}$, which verifies (5). $\square$

4 Main Result

Theorem 12 *Let Φ be a non-degenerated Musielak-Orlicz function, $h_{\Phi} \cong \ell_{\Phi}$, and $f : \ell_{\Phi} \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and bounded below function with a bounded domain S . Then the set*

$$A = \{a \in \ell_{\infty} : \text{the minimization problem } (f + \tilde{\Phi}_a) \text{ doesn't have a strong minimum on } S\}$$

is σ -porous and $F : \ell_{\infty} \rightarrow \ell_{\Phi}, a \mapsto \text{argmin}_S(f + \tilde{\Phi}_a)$ is singled-valued and upper-semicontinuous on $\ell_{\infty} \setminus A$.

Proof By Lemma 11, since S is bounded, $(\text{span}\{\mathbb{P}\}, \|\cdot\|_{\mathbb{P}})$ is a perturbation space in strong sense for S . We apply first Theorem 7 and obtain that the set

$$A_1 = \left\{ h \in \text{span}\{\mathbb{P}\} : \text{the minimization problem } (f + h, S) \text{ is not WPMC} \right\}$$

is a σ -porous set in $\text{span}\{\mathbb{P}\}$. Since $(\text{span}\{\mathbb{P}\}, \|\cdot\|_{\mathbb{P}})$ is isometric to $(\ell_{\infty}, \|\cdot\|_{\infty})$, the set

$$\tilde{A}_1 = \{a \in \ell_{\infty} : \text{the function } f + \langle a, \tilde{x} \rangle \text{ doesn't attain its minimum on } \tilde{S}\}$$

is σ -porous in ℓ_{∞} . This implies that the mapping F has a domain $\text{Dom}(F) = \ell_{\infty} \setminus \tilde{A}_1$. Now Theorem 6 shows that F and, therefore, $\tilde{F}$ defined by (4) is single-valued and upper-semicontinuous on a set $L := \ell_{\infty} \setminus (\tilde{A}_1 \cup \tilde{A}_2)$, where $\tilde{A}_2$ is σ -fully cone porous in ℓ_{∞} .

The function $\tilde{f}^* : \ell_{\infty} \rightarrow \mathbb{R}$ defined as

$$\tilde{f}^*(a) = \sup_{\tilde{x} \in \tilde{S}} \left\{ \langle a, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \right\} \quad (30)$$

is convex, lower semi-continuous and w^* -lower semi-continuous with domain ℓ_{∞} , because $\tilde{S}$ is bounded.

Therefore, from the basic facts of convex analysis (see for instance [28], Theorem 4.4.2),

$$\tilde{f}^*(a) = \sup_{\tilde{x} \in \ell_1} \left\{ \langle a, \tilde{x} \rangle - \tilde{f}^{**}(\tilde{x}) \right\}, \quad (31)$$

where $\tilde{f}^{**} : \ell_1 \rightarrow \mathbb{R}$ is its conjugate:

$$\tilde{f}^{**}(\tilde{x}) = \sup_{a \in \ell_{\infty}} \left\{ \langle a, \tilde{x} \rangle - \tilde{f}^*(a) \right\}.$$

We will prove that $\tilde{f}^*$ is Fréchet differentiable at every point of $-L$. Let $a \in -L$, so $\tilde{F}(-a)$ is singleton, say $\tilde{x}_a$. Then

$$\begin{aligned} \frac{1}{t} \left[\tilde{f}^*(a + th) - \tilde{f}^*(a) \right] &= \frac{1}{t} \left[\sup_{\tilde{x} \in \tilde{S}} \left\{ \langle a + th, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \right\} - \sup_{\tilde{x} \in \tilde{S}} \left\{ \langle a, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \right\} \right] \\ &\geq \frac{1}{t} \left[\langle a + th, \tilde{x}_a \rangle - \tilde{f}(\tilde{x}_a) - \langle a, \tilde{x}_a \rangle + \tilde{f}(\tilde{x}_a) \right] \\ &= \langle h, \tilde{x}_a \rangle. \end{aligned} \quad (32)$$

Let $h \in B, t \in (0, 1)$ and $d = \sup_{\tilde{x} \in \tilde{S}} \|\tilde{x}\|$. By upper semi-continuity of $\tilde{F}$ at $-a$,

$$\exists \delta > 0 : a + y \in -L, \|y\| < \delta \implies \|\tilde{F}(-a - y) - \tilde{F}(-a)\| < t. \quad (33)$$

Choose

$$y \in B\left(th, \min\left\{\delta, \frac{t^2}{d}\right\}\right) \cap (-a - L)$$

and denote

$$\tilde{y} = \text{argmax}_{\tilde{x} \in \tilde{S}} \left\{ \langle a + y, \tilde{x} \rangle - \tilde{f}^*(\tilde{x}) \right\}.$$

Then

$$\begin{aligned} \frac{1}{t} \left[\tilde{f}^*(a + th) - \tilde{f}^*(a) \right] &= \frac{1}{t} \left[\sup_{\tilde{x} \in \tilde{S}} \left\{ \langle a + y + th - y, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \right\} - \sup_{\tilde{x} \in \tilde{S}} \left\{ \langle a, \tilde{x} \rangle - \tilde{f}(\tilde{x}) \right\} \right] \\ &\leq \frac{1}{t} \left[\sup_{\tilde{x} \in \tilde{S}} \left\{ \langle a + y, \tilde{x} \rangle - \tilde{f}^*(\tilde{x}) \right\} + \sup_{\tilde{x} \in \tilde{S}} \langle th - y, \tilde{x} \rangle - \langle a, \tilde{y} \rangle + \tilde{f}(\tilde{y}) \right] \\ &\leq \frac{1}{t} \langle a + y, \tilde{y} \rangle - \tilde{f}^*(\tilde{y}) + \|th - y\|d - \langle a, \tilde{y} \rangle + \tilde{f}^*(\tilde{y}) \\ &= \frac{1}{t} \left[\langle y, \tilde{y} \rangle + t^2 \right] \\ &= \frac{1}{t} \left[\langle y - th, \tilde{y} \rangle + \langle th, \tilde{y} - \tilde{x}_a \rangle + \langle th, \tilde{x}_a \rangle + t \right] \quad (\text{by (33)}) \end{aligned}$$

$$\leq t/d + t + \langle h, \tilde{x}_a \rangle + t$$

This and (32) shows that

$$\lim_{t \downarrow 0} \frac{1}{t} [\tilde{f}^*(a + th) - \tilde{f}^*(a)] = \langle h, \tilde{x}_a \rangle \quad \text{uniformly on } h \in B,$$

i.e. $\tilde{f}^*$ is Fréchet differentiable at a with $\tilde{f}^{*'}(a) = \tilde{x}_a$.

From another basic fact from convex analysis (see for instance [28], Proposition 4.4.5),

$$\partial \tilde{f}^*(a) = \operatorname{argmax}_{\tilde{x} \in \ell_1} \left\{ \langle a, \tilde{x} \rangle - \tilde{f}^{**}(\tilde{x}) \right\} = \tilde{F}(-a)$$

Let $\tilde{x}_n \subset \tilde{S}$ be a minimizing sequence of the function

$$\langle -a, \tilde{x} \rangle + \tilde{f}(\tilde{x}).$$

Since $\tilde{f}^{**} \leq \tilde{f}$, $\tilde{x}_n \subset \tilde{S}$ is a maximizing sequence for the function

$$\langle a, \tilde{x} \rangle - \tilde{f}^{**}(\tilde{x}).$$

Now Proposition 4 from [29] implies that $\tilde{x}_n$ converges in norm to $\tilde{F}(-a) = F(-a)$ and since $-a$ was an arbitrary point from L , we finish the proof. $\square$

If the domain of f is not bounded, we obtain similar result in the next theorem, however the perturbation space is ℓ_∞^+ .

Theorem 13 *Let Φ be a non-degenerated Musielak-Orlicz function, $h_\Phi \cong \ell_\Phi$, and $f : \ell_\Phi \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous and bounded below function. Then the set*

$$A = \{a \in \ell_\infty^+ : \text{the minimization problem } (f + \tilde{\Phi}_a) \text{ doesn't have a strong minimum on } \ell_\Phi\}$$

is σ -porous and the multivalued mapping $F : \ell_\infty \rightarrow \ell_\Phi, a \mapsto \operatorname{argmin}_{\ell_\Phi} (f + \tilde{\Phi}_a)$ is singled-valued and upper-semicontinuous on $\ell_\infty^+ \setminus A$.

Proof Case 1: f is coercive, i.e.,

$$\lim_{\|x\|_\Phi \rightarrow \infty} f(x) = +\infty. \quad (34)$$

Fix $x_0 \in \operatorname{Dom}(f)$ and $K > \|x_0\|_\Phi$ be so big that from $\|x\|_\Phi \geq K$ it follows that

$$f(x) > f(x_0) + \tilde{\Phi}(x_0). \quad (35)$$

Let us put $S = KB_{\ell_\Phi}$. From Lemma 11, since S is bounded, $\mathbb{P}$ is a perturbation space in a strong sense for S .

By Theorem 12, the set

$$A = \{a \in \ell_\infty^+ : \text{the minimization problem } (f + \tilde{\Phi}_a) \text{ doesn't have a strong minimum on } S\}$$

is σ -porous and $F : \ell_\infty^+ \rightarrow \ell_\Phi, a \mapsto \operatorname{argmin}_S (f + \tilde{\Phi}_a)$ is singled-valued and upper-semicontinuous on $\ell_\infty^+ \setminus A$.

Take any $a \in (\ell_\infty^+ \setminus A)$. Let $z \in S$ be a minimum point of $f + \tilde{\Phi}_a$ on S , i.e., $f(z) + \tilde{\Phi}_a(z) \leq f(x) + \tilde{\Phi}_a(x)$ for any $x \in S$.

If $x \notin S$, then $\|x\| > K$, and

$$\begin{aligned} f(x) + \tilde{\Phi}_a(x) &\geq f(x) && (\text{because } a \geq 0) \\ &\geq f(x_0) + \tilde{\Phi}_a(x_0) && (\text{by (35)}) \\ &\geq f(z) + \tilde{\Phi}_a(z). \end{aligned}$$

So we proved that z is a global minimum of the function $f + \tilde{\Phi}_a$ on ℓ_Φ .

Case 2: f is only bounded from below. For every natural $n \in \mathbb{N}$ put $f_n = f + \frac{1}{n}\tilde{\Phi}$. From $\lim_{\|x\|_\Phi \rightarrow \infty} \tilde{\Phi}(x) = +\infty$ it follows that f_n is coercive, i.e., satisfies (34). By case 1, the set

$$A_n = \left\{ a \in \ell_\infty^+ : \text{the function } f + \frac{1}{n}\tilde{\Phi} + \tilde{\Phi}_a = f + \tilde{\Phi}_{a+\frac{1}{n}\mathbf{1}} \right. \quad (36)$$

$$\left. \text{doesn't have a global minimum on } \ell_\Phi \right\}, \quad (37)$$

where $\mathbf{1} = (1, 1, \dots) \in \ell_\infty$, is σ -porous in ℓ_∞^+ and

$$F : \ell_\infty^+ \rightarrow \ell_\Phi, a \mapsto \operatorname{argmin}_{\ell_\Phi} (f + \tilde{\Phi}_a)$$

is singled-valued and upper-semicontinuous on $\ell_\infty^+ \setminus A$.

The set $B_n = A_n + \frac{1}{n}\mathbf{1}$, is σ -porous too, as a translation of a σ -porous set. Then

$$b \in \left(\ell_\infty^+ + \frac{1}{n}\mathbf{1} \right) \setminus B_n, \implies \text{the function } f + \tilde{\Phi}_b \text{ has a global minimum on } \ell_\Phi. \quad (38)$$

Claim. The set

$$L = \ell_\infty^+ \setminus \bigcup_{n=1}^{\infty} \left(\ell_\infty^+ + \frac{1}{n}\mathbf{1} \right)$$

is porous.

Proof. Take any $B(x, r) \subset \ell_\infty^+$ and define the point $y \in \ell_\infty^+$ by $y_i = x_i + r/2$. Then $B(y, r/4) \subset B(x, r)$ and for any $z \in B(y, r/4)$ we have

$$z_i \geq y_i - r/4 = x_i + r/2 - r/4 \geq r/4.$$

This means $B(y, r/4) \subset \ell_\infty^+ + \frac{1}{n}\mathbf{1}$ for $n > 4/r$, so $B(y, r/4) \cap L = \emptyset$ and the Claim is proved.

The set

$$M = L \bigcup \bigcup_{n=1}^{\infty} \left(A_n + \frac{1}{n}\mathbf{1} \right)$$

is σ -porous.

Let $b \in \ell_\infty^+ \setminus M$. Then $b \in \ell_\infty^+ + \frac{1}{n_0}\mathbf{1}$ for some n_0 and $b \notin A_{n_0} + \frac{1}{n_0}\mathbf{1}$. Now (38) shows that $f + \tilde{\Phi}_b$ has a global minimum on ℓ_Φ and F is single-valued and upper semi-continuous at b , and the proof is completed. $\square$

5 Applications

5.1 The Stegall Variational Principle in Musielak-Orlicz Sequence Spaces

Theorem 14 *Let $X = \ell_\Phi$ be a Musielak-Orlicz sequence space, Φ be a non-degenerated Musielak-Orlicz function, $h_\Phi \cong \ell_\Phi$, and $f : \ell_\Phi \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous and bounded below function, with bounded domain $S := \{x \in \ell_\Phi : f(x) < \infty\}$. Then there exists a dense G_δ subset $X_0^* \in X^*$ such that for every $x^* \in X_0^*$ the function $f - x^*$ attains its strong minimum on X .*

Proof Let $\varepsilon > 0$ be given. Denote

$$K_\varepsilon = \sup_{y \in B(0, \varepsilon)} \sup_{\|x - y\| = 1} \|\tilde{x} - \tilde{y}\|.$$

Take an element

$$a \in (l_\infty^- \setminus A) \cap B(0, \varepsilon/K_\varepsilon),$$

where $l_\infty^- = \{a \in \ell_\infty : a_i \leq 0\}$, A is given by Theorem 12, and let x_0 be the strong minimum of $f + \tilde{\Phi}_a = f - \tilde{\Phi}_{-a}$. Since $\tilde{\Phi}_{-a}$ is convex, the subdifferential $\partial\tilde{\Phi}_{-a}(x_0)$ is non-empty. Take any element $x^* \in \partial\tilde{\Phi}_{-a}(x_0)$ and write:

$$f(x) - f(x_0) \geq \tilde{\Phi}_{-a}(x) - \tilde{\Phi}_{-a}(x_0) \geq \langle x^*, x - x_0 \rangle, \quad (39)$$

for every $x \in X$, which shows that x_0 is a minimum for $f(\cdot) - \langle x^*, \cdot \rangle$.

Let $\{x_n\}_{n=1}^\infty \subset X$ be a minimizing sequence of $f(\cdot) - \langle x^*, \cdot \rangle$, i.e.,

$$\lim_{n \rightarrow \infty} (f(x_n) - \langle x^*, x_n \rangle) = f(x_0) - \langle x^*, x_0 \rangle.$$

From (39) we have:

$$f(x_n) - f(x_0) - \langle x^*, x_n - x_0 \rangle \geq f(x_n) - f(x_0) - (\tilde{\Phi}_{-a}(x_n) - \tilde{\Phi}_{-a}(x_0)) \geq 0,$$

which shows that $\{x_n\}_{n=1}^\infty$ is a minimizing sequence for $f + \tilde{\Phi}_a$, too.

Since x_0 is a strong minimum for $f + \tilde{\Phi}_a$, $\{x_n\}_{n=1}^\infty$ must converge to x_0 .

From (39) we have also

$$\langle x^*, x - x_0 \rangle \leq \langle -a, \tilde{x} - \tilde{x}_0 \rangle \leq \|a\| \|\tilde{x} - \tilde{x}_0\|$$

whence

$$\|x^*\| \leq \|a\| \sup_{\|x - x_0\| = 1} \|\tilde{x} - \tilde{x}_0\| \leq \varepsilon.$$

So we prove the density part, since this reasoning can be done to any linear perturbation of f . The set

$$X_n^* := \left\{ x \in X^* : \exists t > 0 : \text{diam}(\Omega_{f-x^*}^S(t)) < \frac{1}{n} \right\} \quad (40)$$

is dense and open and

$$X_0^* := \bigcap_{n=1}^\infty X_n^*$$

satisfies the conclusion of the theorem. $\square$

Corollary 15 *Let ℓ_Φ be a Musielak-Orlicz sequence space, equipped with the Luxemburg's norm, Φ be a non-degenerated Musielak-Orlicz function, $h_\Phi \cong \ell_\Phi$, then ℓ_Φ is a dentable space, has the Radon-Nikodym property and ℓ_Φ^* is a w^* -Asplund space.*

Proof The assertions follow from Theorem 5 in [29], where the reader can find also definitions of the involved notions. $\square$

Corollary 16 *Let X be a Nakano sequence space, equipped with the Luxemburg's norm, $\Phi = \left\{ \frac{|t|^{p_n}}{p_n} \right\}_{n=1}^\infty$. If $\limsup_{n \rightarrow \infty} p_n = q < +\infty$ then $\ell_{\{p_n\}}$ is a dentable space.*

5.2 Application to the Differentiability of Bump Functions in Musielak-Orlicz Sequence Spaces

Theorem 17 *Let Φ be a Musielak-Orlicz function and $1 < \alpha_\Phi < 2$.*

Then

1. *there is an equivalent Fréchet differentiable norm, and therefore, ℓ_Φ is an Asplund space*
2. *if more over, $h_\Phi \cong \ell_\Phi$ and there are a sequences $\{h^{(n)}\}_{n=1}^\infty \subset S_{\ell_\Phi}$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$, $\lim_{n \rightarrow \infty} t_n = 0$, so that*

$$\lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} \frac{\Phi_k(t_n h_k^{(n)})}{|t_n|^p} = +\infty$$

for some $p \in (1, 2]$ then there is no nontrivial Fréchet differentiable bump function on ℓ_Φ , $b : \ell_\Phi \rightarrow \mathbb{R}$ with and estimate

$$b(x+h) = b(x) + b'(x)(h) + O(\|h\|^p).$$

Proof The existence follows from Theorem 4.

For the second part, suppose the contrary, i.e. there is a bump b with the assumed estimate. Let us put

$$f(x) = \begin{cases} b^{-2}(x), & b(x) \neq 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

It is clear that f is proper, lower semi-continuous function which is positive and Fréchet differentiable at each point of $\text{domain}(f)$ and the latter is a bounded set.

Applying Theorem 12, we get $a \in \ell_\infty$ such that $1 \leq a_n \leq 2$ for all $n \in \mathbb{N}$, and $z \in \text{domain}(f)$ such that $f + \tilde{\Phi}_{-a} = f - \tilde{\Phi}_a$ attains its minimum at z . In particular

$$f(x) - f(z) \geq \tilde{\Phi}_a(x) - \tilde{\Phi}_a(z)$$

for all $z \in \ell_\Phi$.

Consequently we get

$$\frac{f(z+h) + f(z-h) - 2f(z)}{\|h\|^p} \geq \frac{\tilde{\Phi}_a(z+h) + \tilde{\Phi}_a(z-h) - 2\tilde{\Phi}_a(z)}{\|h\|^p}.$$

Letting h to 0, the left hand side of the above is bounded, because f is Fréchet differentiable at z with the desired estimate,

$$\limsup_{\|h\| \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{\|h\|^p} < +\infty$$

for all $x \in \ell_\Phi$. Therefore,

$$\limsup_{\|h\| \rightarrow 0} \frac{\tilde{\Phi}_a(z+h) + \tilde{\Phi}_a(z-h) - 2\tilde{\Phi}_a(z)}{\|h\|^p} < +\infty$$

at the point z .

Let us choose sequences $\{h^{(n)}\}_{n=1}^\infty \subset S_{\ell_\Phi}$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$, $\lim_{n \rightarrow \infty} t_n = 0$, so that

$$\lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} \frac{\Phi_k(t_n h_k^{(n)})}{|t_n|^p} = +\infty.$$

We can write the chain of inequalities

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{\tilde{\Phi}_a(z + t_n h^{(n)}) + \tilde{\Phi}_a(z - t_n h^{(n)}) - 2\tilde{\Phi}_a(z)}{\|t_n h^{(n)}\|^p} \\
& \geq \lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} a_k \frac{\Phi_k(z_n + t_n h_k^{(n)}) + \Phi_k(z_n - t_n h_k^{(n)}) - 2\Phi_k(z_n)}{|t_n|^p} \\
& \geq 2 \sup_{k \in \mathbb{N}} \{a_k\} \lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} \frac{\Phi_k(t_n h_k^{(n)})}{|t_n|^p} \\
& \geq 2 \lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} \frac{\Phi_k(t_n h_k^{(n)})}{|t_n|^p} = +\infty.
\end{aligned}$$

We get a contradiction. $\square$

Corollary 18 *Let $\ell_{\{p_n\}}$ be a Nakano sequence space, $p_n \geq 1$, and $\liminf_{n \rightarrow \infty} p_n > 1$. Then*

- (a) *there is an equivalent Fréchet differentiable norm in $\ell_{\{p_n\}}$ and, therefore, $\ell_{\{p_n\}}$ is an Asplund space*
- (b) *if more over $\limsup_{n \rightarrow \infty} p_n < +\infty$, then on $\ell_{\{p_n\}}$ there is no nontrivial Fréchet differentiable bump function $b : \ell_{\{p_n\}} \rightarrow \mathbb{R}$ with an estimate*

$$b(x + h) = b(x) + b'(x)(h) + O(\|h\|^{p+\varepsilon})$$

for any $\varepsilon > 0$.

Proof (a) follows from Theorem 4.

The Nakano sequence space $\ell_{\{p_n\}}$ is a special case of a Musielak-Orlicz sequence space $\ell_{\tilde{\Phi}}$, with $\Phi_k(t) = t^{p_k}$.

For (b) $\limsup_{n \rightarrow \infty} p_n < +\infty$ is equivalent to $h_{\{p_n\}} \cong \ell_{\{p_n\}}$ and we will apply Theorem 17.

Let us choose a subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ so that $\lim_{k \rightarrow \infty} p_{n_k} = p$ and then choose $h^{(k)} = e_{n_k}$. For any arbitrary chosen $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ so that for any $k \geq k_0$ there holds $p_{n_k} - p - \varepsilon < 0$. Thus, for any arbitrary sequence $\{t_k\}_{k=1}^{\infty}$, satisfying $t_k > 0$ and $\lim_{k \rightarrow \infty} t_k = 0$, we get

$$\lim_{k \rightarrow \infty} \frac{\tilde{\Phi}(t_k h^{(k)})}{|t_k|^{p+\varepsilon}} = \lim_{k \rightarrow \infty} \frac{t_k^{p_{n_k}} |h_{n_k}^{(k)}|^{p_{n_k}}}{|t_k|^{p+\varepsilon}} = \limsup_{k \rightarrow \infty} t_k^{p_{n_k} - p - \varepsilon} = +\infty.$$

$\square$

Corollary 19 *Let M be an Orlicz function, $w = \{w_n\}_{n=1}^{\infty}$ be a wight sequence from the class Λ_{∞} , and $1 < \alpha_M < 2$. Then*

- 1. *there is an equivalent Fréchet differentiable norm and $\ell_M(w)$ is an Asplund space*
- 2. *if more over $h_M(w) \cong \ell_M(w)$ and there are a sequences $\{h^{(n)}\}_{n=1}^{\infty} \subset S_{\ell_M(w)}$ and $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$, $\lim_{n \rightarrow \infty} t_n = 0$, so that*

$$\lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} w_k \frac{M(t_n h_k^{(n)})}{|t_n|^p} = +\infty$$

for some $p \in (1, 2]$, then on $\ell_M(w)$ there is no nontrivial Fréchet differentiable bump function $b : \ell_M(w) \rightarrow \mathbb{R}$ with and estimate

$$b(x+h) = b(x) + b'(x)(h) + O(\|h\|^p).$$

Proof The weighted Orlicz sequence space $\ell_M(w)$, with $w \in \Lambda_\infty$ is a special case of a Musielak-Orlicz sequence space $\ell_{\tilde{\Phi}}$, with $\Phi_k(t) = w_k M(t)$.

Since $h_M(w) \cong \ell_M(w)$, we can apply Theorem 17. $\square$

Corollary 20 *Let M be an Orlicz function, $w = \{w_n\}_{n=1}^\infty$ be a weight sequence from the class Λ , and $1 < \alpha_M < 2$. Then*

1. *there is an equivalent Fréchet differentiable norm*
2. *if moreover, $h_M(w) \cong \ell_M(w)$ and there are a sequences $\{h^{(n)}\}_{n=1}^\infty \subset S_{\ell_M(w)}$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}$, $\lim_{n \rightarrow \infty} t_n = 0$, so that*

$$\lim_{n \rightarrow \infty} \sum_{k \in \text{supp}(h^{(n)})} w_k \frac{M(t_n h_k^{(n)})}{|t_n|^p} = +\infty$$

for some $p \in (1, 2]$, then on $\ell_M(w)$ there is no nontrivial Fréchet differentiable bump function $b : \ell_M(w) \rightarrow \mathbb{R}$ with and estimate

$$b(x+h) = b(x) + b'(x)(h) + O(\|h\|^p).$$

Proof The weighted Orlicz sequence space $\ell_M(w)$, with $w \in \Lambda_\infty$ is a special case of a Musielak-Orlicz sequence space $\ell_{\tilde{\Phi}}$, with $\Phi_k(t) = w_k M(t)$.

Since $h_M(w) \cong \ell_M(w)$, we can apply Theorem 17. $\square$

6 An Illustrative Example of a Weighted Orlicz Sequence Space, generated by on Orlicz function without Δ_2 -Condition

Weighted Orlicz sequence spaces are an important generalization of classical sequence spaces. In order to apply the fundamental results from functional analysis it is necessary the considered weighted Orlicz sequence space to share some of the good geometrical of the Orlicz spaces. Thus we end up with the idea to search for spaces generated by an Orlicz functions without the Δ_2 conditons, but satisfying $\ell_M(w) \equiv h_M(w)$. We will mention just a few spheres of applications of weighted Orlicz sequence spaces: the provide the right function space framework for solutions of certain differential equations with non-standard growth conditions [30]; due to their flexibility in modeling decay and growth of sequences, they can be applied in signal representation [31]; wavelet approximations [32]; Fourier transforms and summability [33].

Let N be an Orlicz function, $\lim_{t \rightarrow 0} \frac{N(t)}{t} = 0$, $N(1) = 1$ and $\{w_k\}_{k=1}^\infty, w_1 = 1$, be a weight sequence from the class Λ_∞ such that $\sum_{k=1}^\infty \frac{w_k}{w_{k+1}} < \infty$.

Denote $a_k = N^{-1}(1/w_k)$ and choose a sequence $\{b_k\}_{k=1}^{\infty}$ fulfilling $0 < a_{k+1} < b_k < a_k < 1$, $\sum_{k=1}^{\infty} \frac{N(b_k)}{N(a_k)} < \infty$, and $\lim_{k \rightarrow \infty} \frac{b_k}{a_k} = 0$.

We define the Orlicz function M by:

$$M(t) = \begin{cases} N(t), & t \in [a_{k+1}, b_k], \quad k \in \mathbb{N} \\ l_k(t), & t \in [b_k, a_k], \quad k \in \mathbb{N}, \end{cases} \quad (41)$$

where the line l_k is defined by $l_k(t) = \frac{N(a_k) - N(b_k)}{a_k - b_k}(t - a_k) + N(a_k)$.

Throughout this paragraph by M we denote the Orlicz function defined in (41), and let $\{a_k\}_{k=1}^{\infty}$ be the sequence defined as $N^{-1}(1/w_k)$.

Let us put $S(t) = t^p M(t)$ for some $p > 0$.

Proposition 21 ([34]) *The weighted Orlicz sequence space $\ell_M(w)$ satisfies $\ell_M(w) \cong h_M(w)$ and*

$$\sum_{k=1}^{\infty} \frac{b_k}{a_k} < \infty.$$

Proposition 22 *Let β_n be the solution of the equation $w_n M(\beta_n) = \beta$ and $w'_n = \frac{w_n}{\beta_n^p}$. The weighted Orlicz sequence space $\ell_S(w')$ satisfies $\ell_S(w') \cong h_S(w')$ if and only if*

$$\sum_{k=1}^{\infty} \frac{b_k}{a_k} < \infty.$$

Proof From $\ell_M(w) \cong h_M(w)$ it follows that the $\ell_{\{w_n M\}} \cong h_{\{w_n M\}}$. Therefore the Musielak-Orlicz function $\{w_n M\}_{n=1}^{\infty}$ has the δ_2 condition at zero. Thus there are $\beta > 0$ and $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < +\infty$ and there holds the inequality

$$w_n M(2t) \leq K w_n M(t) + c_n, \quad (42)$$

provided $t \in [0, \beta_n]$, where $\beta = w_n M(\beta_k)$.

From the chain of inequalities

$$\begin{aligned} \frac{w_n}{\beta_n^p} S(2t) &= \frac{w_n}{\beta_n^p} 2^p t^p M(2t) \leq \frac{2^p t^p}{\beta_n^p} (K w_n M(t) + c_n) \\ &= 2^p K \frac{w_n}{\beta_n^p} t^p M(t) + K c_n = 2^p K \frac{w_n}{\beta_n^p} S(t) + K c_n \\ &= C \frac{w_n}{\beta_n^p} S(t) + b_n, \end{aligned}$$

for $t \in [0, \beta_n]$, where $\beta = w_n M(\beta_k)$, and $\{b_n\}_{n=1}^{\infty} \in \ell_1$ we conclude that the Musielak-Orlicz function $\{w'_n S(t)\}_{n=1}^{\infty}$ has the δ_2 condition at zero and therefore $\ell_S(w') \cong h_S(w')$. $\square$

Proposition 23 *Let N be a non degenerated Orlicz function, $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ and M be defined with respect to (41), and $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \infty$.*

Then $\alpha_M = 1$.

If in addition $\mu a_{k+1} = b_{k+1}$ for some $\mu > 0$ and $N \notin \Delta_2$ then $\beta_M = +\infty$

Proof Let us define $\lambda_k = a_k$ and $t_k = \frac{\gamma_k a_k + (1 - \gamma_k)b_k}{a_k}$, for $\gamma_k \in [0, 1]$. Then,

$$\begin{aligned} \frac{M(\lambda_k t_k)}{t_k^p M(\lambda_k)} &= \frac{(1 - \gamma_k)M(b_k) + \gamma_k M(a_k)}{M(a_k)} \left(\frac{\gamma_k a_k + (1 - \gamma_k)b_k}{a_k} \right)^{-p} \\ &\geq \left(\frac{\gamma_k^{1-1/p} a_k + \gamma_k^{-1/p} (1 - \gamma_k)b_k}{a_k} \right)^{-p}. \end{aligned} \quad (43)$$

We can calculate that if $a < p(a - b)$, and $0 < b < a$, then $0 < \frac{b}{(p-1)(a-b)} < 1$. By assumption $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \infty$. Therefore, for every $p > 1$, there is $N_p \in \mathbb{N}$, so that $0 < \frac{b_k}{(p-1)(a_k - b_k)} < 1$, for any $k > N_p$. Let us substitute $\gamma_k = \frac{b_k}{(p-1)(a_k - b_k)}$ in (43), for $k > N_p$. Then, we obtain $\frac{M(\lambda_k t_k)}{t_k^p M(\lambda_k)} \geq C_p \frac{a_k^p b_k^{1-p}}{a_k - b_k} \geq C_p \left(\frac{a_k}{b_k} \right)^{p-1}$, where $C_p = (p-1)^{p-1} p^{-p}$. Then from $\lim_{k \rightarrow \infty} \frac{a_k}{b_k} = \infty$, it follows that $\lim_{k \rightarrow \infty} \frac{M(\lambda_k t_k)}{t_k^p M(\lambda_k)} = \infty$ for every $p > 1$. Therefore $\alpha_M \leq 1$, but M is a convex function, thus $\alpha_M = 1$.

Let $\mu > 1$ be so that $\mu a_{k+1} = b_{k+1}$. Thus $a_{k+1}, \mu a_{k+1} \in [a_{k+1}, b_{k+1}]$. Then from $N \notin \Delta_2$ and

$$\frac{M(\mu a_{k+1})}{M(a_{k+1})} = \frac{N(\mu a_{k+1})}{N(a_{k+1})}$$

it follows that $M \notin \Delta_2$ and consequently $\beta_M = +\infty$. $\square$

The considered Musielak-Orlicz sequence spaces in [34], Proposition 21 do not satisfy $\alpha_M \in (1, 2)$. Therefore we will slightly change the function M .

Proposition 24 *Let φ and ψ be two Orlicz functions, such that $\varphi(t) = t^q \psi(t)$, for some $q > 0$. Then $\alpha_\varphi = \alpha_\psi + q$ and $\beta_\varphi = \beta_\psi + q$*

Proof From

$$\frac{\varphi(\lambda t)}{t^p \varphi(\lambda)} = \frac{\lambda^q t^q \psi(\lambda t)}{t^p \lambda^q \psi(\lambda)} = \frac{\psi(\lambda t)}{t^{p-q} \psi(\lambda)}.$$

we get the proof. $\square$

Example 2 Let $N(t) = t^2 e^{-\frac{1}{2t}}$ and $w_k = \frac{1}{N(1/k^{2k})}$, $k \in \mathbb{N}$. We define the sequences $\{a_k\}_{k=1}^\infty$, $\{b_k\}_{k=1}^\infty$ by $a_k = \frac{1}{k^{2k}}$, $b_k = \frac{2}{(k+1)^{2(k+1)}}$.

From Proposition 21 it follows that $\ell_M(w) \cong h_M(w)$ and therefore there is β so that the Musielak-Orlicz functions $\Phi_n(t) = w_n M(t)$ satisfy (42) for $t \in [0, \Phi_n^{-1}(\beta)]$.

Let us choose $p \in (0, 1)$ and put $w'_n = \frac{w_n}{\beta_n^p}$, where β_n are the solutions of the equations $w_n M(\beta_n) = \beta$. Let put $S(t) = t^p M(t)$.

We get a weighted Orlicz sequence space $\ell_S(w')$, such that by Proposition 22 $\ell_S(w') \cong h_S(w')$, and by Proposition 24 $\alpha_S = \alpha_M + p = p + 1 < 2$, $\beta_S = \beta_M = +\infty$.

Therefore there is an equivalent Fréchet differentiable norm and there is no nontrivial Fréchet differentiable bump function $b : \ell_M(w) \rightarrow \mathbb{R}$ with and estimate

$$b(x + h) = b(x) + b'(x)(h) + O(\|h\|^{1+p}).$$

7 Discussion

There are results about the geometry of weighted Orlicz sequence spaces [12, 13, 18, 22, 34–37]. Unfortunately when the dual of an Orlicz weighted sequence space is needed there no constructive results about the conjugated Musielak-Orlicz function splitted into an Orlicz functions and a weight sequence. Even in the simplest example, where $M(t) = |t|^p$ the conjugated Musielk-Orlicz function $\Phi_n^* = \sup\{st - w_n M(s) : s \geq 0\}$, we get $\Phi_n^*(t) = \frac{p-1}{p^q w_n^{q-1}} |t|^q$. It will be difficult to get sufficient conditions for simultaneously holding $\ell_M(w) \cong h_M(w)$ and $\ell_{\Phi^*} \cong h_{\Phi^*}$.

An interesting open problems will be to get similar results to those in [30–33] for the classes of investigate in this manuscript weighted Orlicz sequence spaces.

8 Conclusion

We proved that the set of perturbations for which a perturbed lower semi-continuous function is WPMC contains not only dense G_δ set but is also a complement to a σ -porous set in the entire space of perturbations. In such a way we generalize an abstract variational principle in Banach spaces [1]. Furthermore, the stronger idea of Tikhonov well posedness takes the role of WPMC if the space is a Musielak-Orlicz sequence space satisfying $\ell_\Phi \cong h_\Phi$. Additionally the multi-valued mapping assigning a parameter to the solution set is single-valued and upper semi-countinuous on the complement of a σ -porous set. We provide a number of applications. The first is that by demonstrating the applicability of Stegall's variational principle, the Musielak-Orlicz sequence spaces are dentable and possess the Radon-Nikodym property. Consequently, their dual spaces are w^* -Asplund spaces. We established a sufficient condition under which the sequence spaces of Musielak-Orlicz and Nakano type are Asplund ones. We presented applications assessing the type of smoothness of specific Musielak-Orlicz, Nakano, and weighted Orlicz sequence spaces. We show that an Orlicz function can be considered without the Δ_2 condition by selecting a certain weighted sequence $\{w_n\}_{n=1}^\infty$ to obtain $\ell_M(w) \cong h_M(w)$ and apply the main conclusions.

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